

INTEGRAL INEQUALITIES FOR MAPPINGS WHOSE DERIVATIVES ARE s -CONVEX IN THE SECOND SENSE AND APPLICATIONS TO SPECIAL MEANS FOR POSITIVE REAL NUMBERS

MEVLÜT TUNÇ♣ AND SEVİL BALGEÇTİ♠

ABSTRACT. In this paper, the authors establish a new type integral inequalities for differentiable s -convex functions in the second sense. By the well-known Hölder inequality and power mean inequality, they obtain some integral inequalities related to the s -convex functions and apply these inequalities to special means for positive real numbers.

1. INTRODUCTION

1.1. Definitions.

Definition 1. [10] A function $\varphi : I \rightarrow \mathbb{R}$ is said to be convex on I if inequality

$$(1.1) \quad \varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$. We say that φ is concave if $(-\varphi)$ is convex.

Definition 2. [8] Let $s \in (0, 1]$. A function $\varphi : (0, \infty] \rightarrow [0, \infty]$ is said to be s -convex in the second sense if

$$(1.2) \quad \varphi(tx + (1-t)y) \leq t^s \varphi(x) + (1-t)^s \varphi(y),$$

for all $x, y \in (0, b]$ and $t \in [0, 1]$. This class of s -convex functions is usually denoted by K_s^2 .

Certainly, s -convexity means just ordinary convexity when $s = 1$.

1.2. Theorems.

Theorem 1. The Hermite-Hadamard inequality: Let $\varphi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function and $u, v \in I$ with $u < v$. The following double inequality:

$$(1.3) \quad \varphi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v \varphi(x) dx \leq \frac{\varphi(u) + \varphi(v)}{2}$$

is known in the literature as Hadamard's inequality (or Hermite-Hadamard inequality) for convex functions. If φ is a positive concave function, then the inequality is reversed.

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♣Corresponding Author.

Theorem 2. [6] Suppose that $\varphi : [0, \infty) \rightarrow [0, \infty)$ is an s -convex function in the second sense, where $s \in (0, 1]$, and let $a, b \in [0, \infty)$, $a < b$. If $\varphi \in L_1([0, 1])$, then the following inequalities hold:

$$(1.4) \quad 2^{s-1} \varphi\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u} \int_u^v \varphi(x) dx \leq \frac{\varphi(u) + \varphi(v)}{s+1}.$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.4). The above inequalities are sharp. If φ is an s -concave function in the second sense, then the inequality is reversed.

Theorem 3. Let $\varphi : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$\frac{1}{v-u} \int_u^v \varphi(x) dx \leq \frac{1}{2} \left[\varphi\left(\frac{u+v}{2}\right) + \frac{\varphi(u) + \varphi(v)}{2} \right]$$

is known as **Bullen's inequality** for convex functions [5, p.39].

In [4], Dragomir and Agarwal obtained inequalities for differentiable convex mappings which are connected to Hadamard's inequality, as follow:

Theorem 4. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° , where $a, b \in I$, with $a < b$. If $|f'|^q$ is convex on $[a, b]$, then the following inequality holds:

$$(1.5) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} [|f'(a)| + |f'(b)|].$$

In [11], Pearce and Pečarić obtained inequalities for differentiable convex mappings which are connected with Hadamard's inequality, as follow:

Theorem 5. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be differentiable mapping on I° , where $a, b \in I$, with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for some $q \geq 1$, then the following inequality holds:

$$(1.6) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}.$$

If $|f'|^q$ is concave on $[a, b]$ for some $q \geq 1$, then

$$(1.7) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{4} \left| f'\left(\frac{a+b}{2}\right) \right|.$$

In [1], Alomari, Darus and Kirmacı obtained inequalities for differentiable s -convex and concave mappings which are connected with Hadamard's inequality, as follow:

Theorem 6. Let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be differentiable mapping on I° such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|^q$, $q \geq 1$ is concave on $[a, b]$ for some fixed $s \in (0, 1]$, then the following inequality holds:

$$(1.8) \quad \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \left(\frac{b-a}{4} \right) \left(\frac{q-1}{2q-1} \right)^{1-\frac{1}{q}} \left[\left| f'\left(\frac{3a+b}{4}\right) \right| + \left| f'\left(\frac{a+3b}{4}\right) \right| \right].$$

In [12], Tunç and Balgeçti obtained inequalities for differentiable convex functions which are connected with a new type integral inequality, as follow:

Lemma 1. *Let $f : J \rightarrow \mathbb{R}$ be a differentiable function on J° . If $f' \in L[a, b]$, then*

$$(1.9) \quad \begin{aligned} & \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2} \left(\frac{bf(a) - af(b)}{b-a} + f\left(\frac{a+b}{2}\right) \right) \\ &= \frac{1}{4} \int_0^1 (tb + (1-t)a) f' \left(\frac{1-t}{2}b + \frac{1+t}{2}a \right) dt \\ & \quad + \frac{1}{4} \int_0^1 (ta + (1-t)b) f' \left(\frac{1-t}{2}a + \frac{1+t}{2}b \right) dt \end{aligned}$$

for each $t \in [0, 1]$ and $a, b \in J$.

Theorem 7. [12] *Let $f : J \rightarrow \mathbb{R}$ be a differentiable function on J° . If $|f'|$ is convex on J , then*

$$(1.10) \quad \begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2} \left(\frac{bf(a) - af(b)}{b-a} + f\left(\frac{a+b}{2}\right) \right) \right| \\ & \leq \left(\frac{5}{48}a + \frac{7}{48}b \right) |f'(a)| + \left(\frac{7}{48}a + \frac{5}{48}b \right) |f'(b)| \end{aligned}$$

for each $a, b \in J$.

Theorem 8. [12] *Let $f : J \rightarrow \mathbb{R}$ be a differentiable function on J° . If $|f'|^q$ is convex on $[a, b]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$(1.11) \quad \begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2} \left(\frac{bf(a) - af(b)}{b-a} + f\left(\frac{a+b}{2}\right) \right) \right| \\ & \leq \frac{1}{4^{1+1/q}} L_p(a, b) \left[[|f'(b)|^q + 3|f'(a)|^q]^{\frac{1}{q}} + [|f'(a)|^q + 3|f'(b)|^q]^{\frac{1}{q}} \right] \end{aligned}$$

Theorem 9. [12] *Let $f : J \rightarrow \mathbb{R}$ be a differentiable function on J° . If $|f'|^q$ is convex on $[a, b]$ and $q \geq 1$, then*

$$(1.12) \quad \begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2} \left(\frac{bf(a) - af(b)}{b-a} + f\left(\frac{a+b}{2}\right) \right) \right| \\ & \leq \frac{A^{1-\frac{1}{q}}(a, b)}{4 \times 12^{\frac{1}{q}}} \left\{ [|f'(b)|^q (2a+b) + |f'(a)|^q (4a+5b)]^{\frac{1}{q}} \right. \\ & \quad \left. + [|f'(a)|^q (a+2b) + |f'(b)|^q (5a+4b)]^{\frac{1}{q}} \right\} \end{aligned}$$

For recent results and generalizations concerning Hadamard's inequality and concepts of convexity and s -convexity see [1]-[12] and the references therein.

Throughout this paper we will use the following notations and conventions. Let $J = [0, \infty) \subset \mathbb{R} = (-\infty, +\infty)$, and $u, v \in J$ with $0 < u < v$ and $f' \in L[u, v]$ and

$$A(u, v) = \frac{u+v}{2}, \quad G(u, v) = \sqrt{uv},$$

$$L_p(u, v) = \left(\frac{v^{p+1} - u^{p+1}}{(p+1)(v-u)} \right)^{1/p}, \quad u \neq v, \quad p \in \mathbb{R}, \quad p \neq -1, 0$$

be the arithmetic mean, geometric mean, generalized logarithmic mean for $u, v > 0$ respectively.

2. INEQUALITIES FOR s -CONVEX FUNCTIONS AND APPLICATIONS

Theorem 10. *Let $f : J \rightarrow \mathbb{R}$ be a differentiable function on J° . If $|f'|$ is s -convex on $[a, b]$ for some fixed $s \in (0, 1]$, then*

$$(2.1) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2} \left(\frac{bf(a) - af(b)}{b-a} + f\left(\frac{a+b}{2}\right) \right) \right| \\ \leq \frac{b(s2^{s+1} + s + 2) + a(2^{s+2} - s - 2)}{2^{s+2}(s+1)(s+2)} |f'(a)| \\ + \frac{a(s2^{s+1} + s + 2) + b(2^{s+2} - s - 2)}{2^{s+2}(s+1)(s+2)} |f'(b)|$$

for each $x \in [a, b]$.

Proof. Using Lemma 1 and from properties of modulus, and since $|f'|$ is s -convex on J , then we obtain

$$\left| \frac{1}{b-a} \int_a^b f(x) dx + \frac{af(b) - bf(a)}{2(b-a)} - \frac{1}{2} f\left(\frac{a+b}{2}\right) \right| \\ \leq \frac{1}{4} \int_0^1 (tb + (1-t)a) \left| f'\left(\left(\frac{1-t}{2}\right)^s b + \left(\frac{1+t}{2}\right)^s a\right) \right| dt \\ + \frac{1}{4} \int_0^1 (ta + (1-t)b) \left| f'\left(\left(\frac{1-t}{2}\right)^s a + \left(\frac{1+t}{2}\right)^s b\right) \right| dt \\ \leq \frac{1}{4} \int_0^1 (tb + (1-t)a) \left[\left(\frac{1-t}{2}\right)^s |f'(b)| + \left(\frac{1+t}{2}\right)^s |f'(a)| \right] dt \\ + \frac{1}{4} \int_0^1 (ta + (1-t)b) \left[\left(\frac{1-t}{2}\right)^s |f'(a)| + \left(\frac{1+t}{2}\right)^s |f'(b)| \right] dt \\ \leq \frac{1}{4} \int_0^1 (tb + (1-t)a) \left(\frac{1-t}{2}\right)^s |f'(b)| dt + \frac{1}{4} \int_0^1 (tb + (1-t)a) \left(\frac{1+t}{2}\right)^s |f'(a)| dt \\ + \frac{1}{4} \int_0^1 (ta + (1-t)b) \left(\frac{1-t}{2}\right)^s |f'(a)| dt + \frac{1}{4} \int_0^1 (ta + (1-t)b) \left(\frac{1+t}{2}\right)^s |f'(b)| dt \\ = \frac{1}{2^{s+2}} \frac{as + a + b}{(s+1)(s+2)} |f'(b)| + \frac{1}{2^{s+2}} \frac{b(s2^{s+1} + 1) + a(2^{s+2} - s - 3)}{(s+1)(s+2)} |f'(a)| \\ + \frac{1}{2^{s+2}} \frac{bs + b + a}{(s+1)(s+2)} |f'(a)| + \frac{1}{2^{s+2}} \frac{a(s2^{s+1} + 1) + b(2^{s+2} - s - 3)}{(s+1)(s+2)} |f'(b)|$$

$$\begin{aligned}
&= \frac{1}{2^{s+2}} \left(\frac{as + a + b}{(s+1)(s+2)} + \frac{a(s2^{s+1} + 1) + b(2^{s+2} - s - 3)}{(s+1)(s+2)} \right) |f'(b)| \\
&\quad + \frac{1}{2^{s+2}} \left(\frac{bs + b + a}{(s+1)(s+2)} + \frac{b(s2^{s+1} + 1) + a(2^{s+2} - s - 3)}{(s+1)(s+2)} \right) |f'(a)| \\
&= \frac{1}{2^{s+2}} \left(\frac{a(s2^{s+1} + s + 2) + b(2^{s+2} - s - 2)}{(s+1)(s+2)} \right) |f'(b)| \\
&\quad + \frac{1}{2^{s+2}} \left(\frac{b(s2^{s+1} + s + 2) + a(2^{s+2} - s - 2)}{(s+1)(s+2)} \right) |f'(a)|
\end{aligned}$$

□

Proposition 1. Let $a, b \in J^\circ$, $0 < a < b$ and $s \in (0, 1]$ then

$$\begin{aligned}
&\left| L_s^s(a, b) + \frac{(s-1)G^2(a, b)L_{s-2}^{s-2}(a, b) - A^s(a, b)}{2} \right| \\
(2.2) \quad &\leq \frac{s[(ab^{s-1} + a^{s-1}b)(s2^{s+1} + s + 2) + (a^s + b^s)(2^{s+2} - s - 2)]}{2^{s+2}(s+1)(s+2)}
\end{aligned}$$

Proof. The proof follows from (2.1) applied to the s -convex function $f(x) = x^s$ and $|f'(x)| = sx^{s-1}$. □

Proposition 2. Let $a, b \in J^\circ$, $0 < a < b$, $s \in (0, 1)$ then

$$\begin{aligned}
(2.3) \quad &\left| \frac{L_s^s(a, b)}{1-s} - \frac{sG^2(a, b)L_{-1-s}^{-1-s}(a, b) + A^{1-s}(a, b)}{2(1-s)} \right| \\
&\leq \frac{1}{2^{s+2}b^s} \left(\frac{a(s2^{s+1} + s + 2) + b(2^{s+2} - s - 2)}{(s+1)(s+2)} \right) \\
&\quad + \frac{1}{2^{s+2}a^s} \left(\frac{a(s - 2^{s+2} + 2) - b(s2^{s+1} + s + 2)}{(s+1)(s+2)} \right)
\end{aligned}$$

Proof. The proof follows from (2.1) applied to the s -convex function $f(x) = \frac{x^{1-s}}{1-s}$ and $|f'(x)| = 1/x^s$ with $s \in (0, 1)$. □

Remark 1. In (2.1), (2.2), if we take $s \rightarrow 1$, then (2.1), (2.2) reduces to (1.10), [12, Proposition 2], respectively.

Theorem 11. Let $f : J \rightarrow \mathbb{R}$ be a differentiable function on J° . If $|f'|$ is s -convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned}
(2.4) \quad &\left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2} \left(\frac{bf(a) - af(b)}{b-a} + f\left(\frac{a+b}{2}\right) \right) \right| \\
&\leq \frac{L_p(a, b)}{4(2^s(s+1))^{\frac{1}{q}}} \left\{ (|f'(b)|^q + (2^{s+1} - 1)|f'(a)|^q)^{\frac{1}{q}} \right. \\
&\quad \left. + (|f'(a)|^q + (2^{s+1} - 1)|f'(b)|^q)^{\frac{1}{q}} \right\}
\end{aligned}$$

for each $x \in [a, b]$.

Proof. Using Lemma 1 and from properties of modulus, and since $|f'|$ is s -convex on J , then we obtain

(2.5)

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx + \frac{af(b) - bf(a)}{2(b-a)} - \frac{1}{2} f\left(\frac{a+b}{2}\right) \right| \\ & \leq \frac{1}{4} \int_0^1 (tb + (1-t)a) \left| f' \left(\left(\frac{1-t}{2} \right)^s b + \left(\frac{1+t}{2} \right)^s a \right) \right| dt \\ & \quad + \frac{1}{4} \int_0^1 (ta + (1-t)b) \left| f' \left(\left(\frac{1-t}{2} \right)^s a + \left(\frac{1+t}{2} \right)^s b \right) \right| dt \end{aligned}$$

Since $|f'|^q$ is s -convex, by the Hölder inequality, we have

$$\begin{aligned} (2.6) \quad & \int_0^1 \left| f' \left(\left(\frac{1-t}{2} \right)^s b + \left(\frac{1+t}{2} \right)^s a \right) \right|^q dt \\ & \leq \int_0^1 \left(\left(\frac{1-t}{2} \right)^s |f'(b)|^q + \left(\frac{1+t}{2} \right)^s |f'(a)|^q \right) dt \end{aligned}$$

and

$$\begin{aligned} (2.7) \quad & \frac{1}{4} \int_0^1 (tb + (1-t)a) \left| f' \left(\left(\frac{1-t}{2} \right)^s b + \left(\frac{1+t}{2} \right)^s a \right) \right| dt \\ & \leq \frac{1}{4} \left(\int_0^1 (tb + (1-t)a)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f' \left(\left(\frac{1-t}{2} \right)^s b + \left(\frac{1+t}{2} \right)^s a \right) \right|^q dt \right)^{\frac{1}{q}} \\ & \quad + \frac{1}{4} \left(\int_0^1 (ta + (1-t)b)^p dt \right)^{\frac{1}{p}} \left(\int_0^1 \left| f' \left(\left(\frac{1-t}{2} \right)^s a + \left(\frac{1+t}{2} \right)^s b \right) \right|^q dt \right)^{\frac{1}{q}} \\ & \leq \frac{1}{4} \left(\int_0^1 (tb + (1-t)a)^p dt \right)^{\frac{1}{p}} \left[\int_0^1 \left(\left(\frac{1-t}{2} \right)^s |f'(b)|^q + \left(\frac{1+t}{2} \right)^s |f'(a)|^q \right) dt \right]^{\frac{1}{q}} \\ & \quad + \frac{1}{4} \left(\int_0^1 (ta + (1-t)b)^p dt \right)^{\frac{1}{p}} \left[\int_0^1 \left(\left(\frac{1-t}{2} \right)^s |f'(a)|^q + \left(\frac{1+t}{2} \right)^s |f'(b)|^q \right) dt \right]^{\frac{1}{q}}. \end{aligned}$$

It can be easily seen that

$$(2.8) \quad \int_0^1 (tb + (1-t)a)^p dt = \int_0^1 (ta + (1-t)b)^p dt = \frac{b^{p+1} - a^{p+1}}{(b-a)(p+1)} = L_p^p(a, b)$$

If expressions (2.6)-(2.8), we obtain

$$\begin{aligned} & \left| \frac{1}{b-a} \int_a^b f(x) dx + \frac{af(b) - bf(a)}{2(b-a)} - \frac{1}{2} f\left(\frac{a+b}{2}\right) \right| \\ & = \frac{1}{4} L_p(a, b) \left[\frac{1}{2^s(s+1)} (|f'(b)|^q + (2^{s+1} - 1) |f'(a)|^q) \right]^{\frac{1}{q}} \\ & \quad + \frac{1}{4} L_p(a, b) \left[\frac{1}{2^s(s+1)} (|f'(a)|^q + (2^{s+1} - 1) |f'(b)|^q) \right]^{\frac{1}{q}}. \end{aligned}$$

The proof is completed. \square

Proposition 3. Let $a, b \in J^\circ$, $0 < a < b$ and $s \in (0, 1]$, then

$$(2.9) \quad \left| L_s^s(a, b) + \frac{(s-1)G^2(a, b)L_{s-2}^{s-2}(a, b) - A^s(a, b)}{2} \right| \\ \leq \frac{L_p(a, b)}{(2^{2q+s}(s+1))^{\frac{1}{q}}} \left\{ \left((sb^{s-1})^q + (2^{s+1} - 1)(sa^{s-1})^q \right)^{\frac{1}{q}} \right. \\ \left. + \left((sa^{s-1})^q + (2^{s+1} - 1)(sb^{s-1})^q \right)^{\frac{1}{q}} \right\}.$$

Proof. The proof follows from (2.4) applied to the s -convex function $f(x) = x^s$ and $|f'(x)| = sx^{s-1}$. \square

Proposition 4. Let $a, b \in J^\circ$, $0 < a < b$ and $s \in (0, 1)$, then

$$(2.10) \quad \left| \frac{L_s^s(a, b)}{1-s} - \frac{sG^2(a, b)L_{-1-s}^{-1-s}(a, b) + A^{1-s}(a, b)}{2(1-s)} \right| \\ \leq \frac{L_p(a, b)}{(2^{2q+s}(s+1))^{\frac{1}{q}}} \left\{ (b^{-sq} + (2^{s+1} - 1)a^{-sq})^{\frac{1}{q}} \right. \\ \left. + (a^{-sq} + (2^{s+1} - 1)b^{-sq})^{\frac{1}{q}} \right\}.$$

Proof. The proof follows from (2.4) applied to the s -convex function $f(x) = \frac{x^{1-s}}{1-s}$ and $|f'(x)| = 1/x^s$. \square

Remark 2. In (2.4), (2.9), if we take $s \rightarrow 1$, then (2.4), (2.9) reduces to (1.11), [12, Proposition 5], respectively.

Theorem 12. Let $f : J \rightarrow \mathbb{R}$ be a differentiable function on J° . If $|f'|$ is s -convex on $[a, b]$ for some fixed $s \in (0, 1]$ and $q > 1$, then

$$(2.11) \quad \left| \frac{1}{b-a} \int_a^b f(x) dx - \frac{1}{2} \left(\frac{bf(a) - af(b)}{b-a} + f\left(\frac{a+b}{2}\right) \right) \right| \\ \leq \frac{A^{1-\frac{1}{q}}(a, b)}{(2^{2q+s}(s+1)(s+2))^{\frac{1}{q}}} \times \\ \left\{ [(as + a + b)|f'(b)|^q + (b(s2^{s+1} + 1) + a(2^{s+2} - s - 3))|f'(a)|^q]^{\frac{1}{q}} \right. \\ \left. + [(bs + b + a)|f'(a)|^q + (a(s2^{s+1} + 1) + b(2^{s+2} - s - 3))|f'(b)|^q]^{\frac{1}{q}} \right\}$$

Proof. From Lemma 1 and using the well-known power mean inequality and since $|f'|^q$ is s -convex on $[a, b]$, we can write

$$\begin{aligned}
& \left| \frac{1}{b-a} \int_a^b f(x) dx + \frac{af(b) - bf(a)}{2(b-a)} - \frac{1}{2} f\left(\frac{a+b}{2}\right) \right| \\
& \leq \frac{1}{4} \left(\int_0^1 (tb + (1-t)a) dt \right)^{1-\frac{1}{q}} \left[\int_0^1 (tb + (1-t)a) \left| f'\left(\frac{1-t}{2}b + \frac{1+t}{2}a\right) \right|^q dt \right]^{\frac{1}{q}} \\
& \quad + \frac{1}{4} \left(\int_0^1 (ta + (1-t)b) dt \right)^{1-\frac{1}{q}} \left[\int_0^1 (ta + (1-t)b) \left| f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) \right|^q dt \right]^{\frac{1}{q}} \\
& \leq \frac{1}{4} \left(\int_0^1 (tb + (1-t)a) dt \right)^{1-\frac{1}{q}} \left[\int_0^1 (tb + (1-t)a) \left(\left(\frac{1-t}{2}\right)^s |f'(b)|^q + \left(\frac{1+t}{2}\right)^s |f'(a)|^q \right) dt \right]^{\frac{1}{q}} \\
& \quad + \frac{1}{4} \left(\int_0^1 (ta + (1-t)b) dt \right)^{1-\frac{1}{q}} \left[\int_0^1 (ta + (1-t)b) \left(\left(\frac{1-t}{2}\right)^s |f'(a)|^q + \left(\frac{1+t}{2}\right)^s |f'(b)|^q \right) dt \right]^{\frac{1}{q}} \\
& \leq \frac{1}{4} \left(\frac{a+b}{2} \right)^{1-\frac{1}{q}} \left[|f'(b)|^q \int_0^1 \left(\frac{1-t}{2}\right)^s (tb + (1-t)a) dt + |f'(a)|^q \int_0^1 \left(\frac{1+t}{2}\right)^s (tb + (1-t)a) dt \right]^{\frac{1}{q}} \\
& \quad + \frac{1}{4} \left(\frac{a+b}{2} \right)^{1-\frac{1}{q}} \left[|f'(a)|^q \int_0^1 \left(\frac{1-t}{2}\right)^s (ta + (1-t)b) dt + |f'(b)|^q \int_0^1 \left(\frac{1+t}{2}\right)^s (ta + (1-t)b) dt \right]^{\frac{1}{q}} \\
& \leq \frac{1}{4} A^{1-\frac{1}{q}}(a, b) \left[\frac{|f'(b)|^q}{2^s} \frac{as + a + b}{(s+1)(s+2)} + \frac{|f'(a)|^q b (s2^{s+1} + 1) + a (2^{s+2} - s - 3)}{(s+1)(s+2)} \right]^{\frac{1}{q}} \\
& \quad + \frac{1}{4} A^{1-\frac{1}{q}}(a, b) \left[\frac{|f'(a)|^q}{2^s} \frac{bs + b + a}{(s+1)(s+2)} + \frac{|f'(b)|^q a (s2^{s+1} + 1) + b (2^{s+2} - s - 3)}{(s+1)(s+2)} \right]^{\frac{1}{q}}.
\end{aligned}$$

The proof is completed. \square

Proposition 5. Let $a, b \in J^\circ$, $0 < a < b$ and $s \in (0, 1]$, then

$$\begin{aligned}
(2.12) \quad & \left| L_s^s(a, b) + \frac{(s-1)G^2(a, b)L_{s-2}^{s-2}(a, b) - A^s(a, b)}{2} \right| \\
& \leq \frac{A^{1-\frac{1}{q}}(a, b)}{(2^{2q+s}(s+1)(s+2))^{\frac{1}{q}}} \times \\
& \quad \left\{ \left[(as + a + b)(sb^{s-1})^q + (b(s2^{s+1} + 1) + a(2^{s+2} - s - 3))(sa^{s-1})^q \right]^{\frac{1}{q}} \right. \\
& \quad \left. + \left[(bs + b + a)(sa^{s-1})^q + (a(s2^{s+1} + 1) + b(2^{s+2} - s - 3))(sb^{s-1})^q \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

Proof. The proof follows from (2.11) applied to the s -convex function $f(x) = x^s$ and $|f'(x)| = sx^{s-1}$. \square

Proposition 6. Let $a, b \in J^\circ$, $0 < a < b$ and $s \in (0, 1]$, then

(2.13)

$$\begin{aligned} & \left| \frac{L_s^s(a, b)}{1-s} - \frac{sG^2(a, b)L_{-1-s}^{-1-s}(a, b) + A^{1-s}(a, b)}{2(1-s)} \right| \\ & \leq \frac{A^{1-\frac{1}{q}}(a, b)}{(2^{2q+s}(s+1)(s+2))^{\frac{1}{q}}} \times \\ & \quad \left\{ [(as + a + b)b^{-sq} + (b(s2^{s+1} + 1) + a(2^{s+2} - s - 3))a^{-sq}]^{\frac{1}{q}} \right. \\ & \quad \left. + [(bs + b + a)a^{-sq} + (a(s2^{s+1} + 1) + b(2^{s+2} - s - 3))b^{-sq}]^{\frac{1}{q}} \right\}. \end{aligned}$$

Proof. The proof follows from (2.11) applied to the s -convex function $f(x) = \frac{x^{1-s}}{1-s}$ and $|f'(x)| = 1/x^s$. \square

Remark 3. In (2.11), (2.12), if we take $s \rightarrow 1$, then (2.11), (2.12) reduces to (1.12), [12, Proposition 8] respectively.

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♣♠ MUSTAFA KEMAL UNIVERSITY, FACULTY OF SCIENCE AND ARTS, DEPARTMENT OF MATHEMATICS, 31000, HATAY, TURKEY

E-mail address: mevluttttunc@gmail.com

E-mail address: sevilbalgeçti@gmail.com